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A differential–difference system related to toroidal Lie algebra

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Abstract
We present a novel differential–difference system in (2+1)-dimensional space–time (one discrete, two continuum), arising from Bogoyavlensky's (2 + 1)-dimensional Korteweg–de Vries hierarchy. Our method is based on the bilinear identity of the hierarchy, which is related to the vertex-operator representation of the toroidal Lie algebra, $\mathfrak{sl}_2^{\mathrm{tor}}$.

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1. Introduction and main results

Multi-dimensional generalization of classical soliton equations is one of the most exciting topics in the field of integrable systems. Among other things, Calogero [1] proposed an interesting example that is a (2 + 1)-dimensional extension of the Korteweg–de Vries (KdV) equation

$$u_t = \tfrac{1}{4}u_{xxy} + uu_y + \tfrac{1}{2}u_x \int^x u_y \, \mathrm{d}x. \tag{1}$$

Yu *et al* [2] obtained multi-soliton solutions of the (2+1)-dimensional KdV equation (1) using Hirota's bilinear method. Let us consider the following Hirota-type equations:

$$(D_x^4 - 4D_x D_{t'})\tau \cdot \tau = 0 \tag{2}$$

$$(D_y D_x^3 + 2D_y D_{t'} - 6D_t D_x)\tau \cdot \tau = 0 \tag{3}$$

where we have used the D-operators of Hirota defined as

$$D_x D_y \ldots f(x, y, \ldots) \cdot g(x, y, \ldots) \stackrel{\text{def}}{=} \partial_s \partial_t f(x + s, y + t, \ldots) g(x - s, y - t, \ldots)|_{s,t,\ldots=0}.$$

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Note that we have introduced the auxiliary variable t' that is a hidden parameter in (1). If we set $u = 2(\log\tau)_{xx}$ and use (2) to eliminate $\partial_{t'}$, then one can show that $u = u(x, y, t)$ solves (1).

Bogoyavlensky [3] showed that there is a hierarchy of higher-order integrable equations associated with (1). In [4], Ikeda and Takasaki generalized Bogoyavlensky's hierarchy from the viewpoint of Sato's theory of KP hierarchy [5–8], and discussed the relationship to toroidal Lie algebras. We note that the relation between integrable hierarchy and toroidal algebras was also discussed by Billig [9] and Iohara *et al* [10] using vertex-operator representations.

In the present paper, we propose the following differential–difference system with the same symmetry:

$$\partial_t u_k = \Delta_{-k}\left(\frac{\partial_x u_{k+1}}{1-\exp(-u_{k+1}-u_k)} - \frac{\partial_x u_k}{1-\exp(u_{k+1}+u_k)} - \frac{1+\exp(u_{k+1}+u_k)}{1-\exp(u_{k+1}+u_k)}v_k\right) \tag{4}$$

$$\Delta_{-k}v_k = \frac{\partial_x u_{k+1}}{u_{k+1}} + \frac{\partial_x(u_{k+1}+u_k)}{1-\exp(u_{k+1}+u_k)} + \frac{\partial_x u_k}{u_k} + \frac{\partial_x(u_k+u_{k-1})}{1-\exp(u_k+u_{k-1})} \tag{5}$$

where Δ_{-k} denotes the backward-difference operator $\Delta_{-k} \stackrel{\text{def}}{=} 1 - \exp(-\partial_k)$ $(\Delta_{-k}u_k = u_k - u_{k-1})$. We also show that this system has soliton-type solutions.

2. Lie algebraic derivation of the bilinear identity

Here we briefly review the Lie algebraic derivation of the bilinear identity of Bogoyavlensky's hierarchy [4], which is a generating function of Hirota-type differential equations. We remark that the Lie algebra considered in [4] is bigger than that considered in this paper. We have not included the derivations to $\mathfrak{sl}_2^{\text{tor}}$ here, since these are not needed for our purpose. Due to this difference, the proof given below may be simpler than that of [4].

The two-toroidal Lie algebra $\mathfrak{sl}_2^{\text{tor}}$ [11, 12] is the universal central extension of the double-loop algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[s, s^{-1}, t, t^{-1}]$, while the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ is the central extension of $\mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}]$. Let A be the ring of Laurent polynomials of two variables s and t. As a vector space, $\mathfrak{sl}_2^{\text{tor}}$ is isomorphic to $\mathfrak{sl}_2 \otimes \mathbb{C}[s, s^{-1}, t, t^{-1}] \oplus \Omega_A/\mathrm{d}A$, where Ω_A denotes the module of Kähler differentials of A defined with the canonical derivation $\mathrm{d} : A \to \Omega_A$. We define the Lie algebra structure of $\mathfrak{sl}_2^{\text{tor}}$ by

$$[x\otimes a, y\otimes b] = [x, y]\otimes ab + (x|y)\overline{(\mathrm{d}a)b} \quad (x, y \in \mathfrak{sl}_2,\ a, b \in A) \tag{6}$$

$$[\mathfrak{sl}_2^{\text{tor}}, \Omega_A/\mathrm{d}A] = 0 \tag{7}$$

where $(x|y)$ denotes the Killing form and $\bar{\cdot} : \Omega_A \to \Omega_A/\mathrm{d}A$ the canonical projection.

In terms of the generating series $X_m(z)$ $(X = E, F, H, m \in \mathbb{Z})$, $K^s_m(z)$ and $K^t_m(z)$, defined by

$$X_m(z) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} X\otimes s^n t^m \cdot z^{-n-1}$$

$$K^s_m(z) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} \overline{s^n t^m\,\mathrm{d}\log s} \cdot z^{-n}$$

$$K^t_m(z) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} \overline{s^n t^m\,\mathrm{d}\log t} \cdot z^{-n-1}$$

the relation (6) can be expressed as

$$X_n(z)Y_n(w) = \frac{1}{z-w}[X, Y]_{m+n}(w) + \frac{1}{(z-w)^2}(X|Y)K^s_{m+n}(w) + \frac{m}{z-w}(X|Y)K^t_{m+n}(w) + \text{regular as } z\to w. \tag{8}$$

There exists a class of representations of $\mathfrak{sl}_2^{\mathrm{tor}}$, which comes directly from that of $\widehat{\mathfrak{sl}}_2$. We consider the space of polynomials

$$F_y \stackrel{\mathrm{def}}{=} \mathbb{C}[y_j,\ j \in \mathbb{Z}] \otimes \mathbb{C}[\exp(\pm y_0)]$$

and define the generating series $\varphi(z)$ and $V_m(y; z)$ by

$$\varphi(z) \stackrel{\mathrm{def}}{=} \sum_{n\in\mathbb{Z}} n y_n z^{n-1} \qquad V_m(y; z) \stackrel{\mathrm{def}}{=} \exp\left[m \sum_{n\in\mathbb{Z}} y_n z^n\right].$$

Proposition 1 (cf [10, 13]). *Let (V, π) be a representation of $\widehat{\mathfrak{sl}}_2$ such that $\overline{\mathrm{d}\log s} \mapsto c \cdot \mathrm{id}_V$ for $c \in \mathbb{C}$. Then we can define the representation π^{tor} of $\mathfrak{sl}_2^{\mathrm{tor}}$ on $V \otimes F_y$ such that*

$$X_m(z) \mapsto X^{\pi}(z) \otimes V_m(z)$$
$$K^s_m(z) \mapsto c \cdot \mathrm{id}_V \otimes V_m(z)$$
$$K^t_m(z) \mapsto c \cdot \mathrm{id}_V \otimes \varphi(z) V_m(z)$$

where $X = E, F, H$, $m \in \mathbb{Z}$ and $X^{\pi}(z) \stackrel{\mathrm{def}}{=} \sum_{n\in\mathbb{Z}} \pi(X \otimes s^n) z^{-n-1}$.

Proof. Using the operator-product expansion for $\widehat{\mathfrak{sl}}_2$

$$X(z)Y(w) = \frac{1}{z-w}[X, Y](w) + \frac{1}{(z-w)^2}(X|Y)K + \text{ regular as } z \to w$$

and the property $V_m(z)V_n(z) = V_{m+n}(z)$, it is straightforward to show that $X_m(z)$ satisfies (8). The remaining relations can be checked by direct calculations. □

To see the relationship to soliton theory, we shall consider the representation of $\widehat{\mathfrak{sl}}_2$ on the fermionic Fock space [7, 8]. Let ψ_j, ψ^*_j $(j \in \mathbb{Z})$ be free fermions with the canonical anti-commutation relation. In terms of the generating series defined as

$$\psi(\lambda) = \sum_{n\in\mathbb{Z}} \psi_n \lambda^n \qquad \psi^*(\lambda) = \sum_{n\in\mathbb{Z}} \psi^*_n \lambda^{-n}$$

the canonical anti-commutation relation is written as

$$[\psi(\lambda), \psi^*(\mu)]_+ = \delta(\lambda/\mu) \qquad [\psi(\lambda), \psi(\mu)]_+ = [\psi^*(\lambda), \psi^*(\mu)]_+ = 0 \tag{9}$$

where $\delta(\lambda) \stackrel{\mathrm{def}}{=} \sum_{n\in\mathbb{Z}} \lambda^n$ is the formal delta-function.

Consider the fermionic Fock space $\mathcal{F}$ with the vacuum vector $|\mathrm{vac}\rangle$ satisfying

$$\psi_j|\mathrm{vac}\rangle = 0 \qquad \text{for} \quad j < 0$$
$$\psi^*_j|\mathrm{vac}\rangle = 0 \qquad \text{for} \quad j \geqslant 0$$

and the dual Fock space $\mathcal{F}^*$ with the dual vacuum vector $\langle\mathrm{vac}|$ satisfying

$$\langle\mathrm{vac}|\psi_j = 0 \qquad \text{for} \quad j \geqslant 0$$
$$\langle\mathrm{vac}|\psi^*_j = 0 \qquad \text{for} \quad j < 0$$
$$\langle\mathrm{vac}|\mathrm{vac}\rangle = 1.$$

As mentioned in [7, 8], a level-1 representation of $\widehat{\mathfrak{sl}}_2$ is given by the elements

$$:\psi(\lambda)\psi^*(-\lambda): = \sum_{j,n\in\mathbb{Z}} (-1)^j :\psi_{j+n}\psi^*_j: \lambda^n$$

where $:\cdot:$ denotes the fermionic normal ordering, $:\psi_i\psi^*_j: \stackrel{\mathrm{def}}{=} \psi_i\psi^*_j - \langle\mathrm{vac}|\psi_i\psi^*_j|\mathrm{vac}\rangle$. Applying proposition 1, we can construct a representation of $\mathfrak{sl}_2^{\mathrm{tor}}$ on the space $\mathcal{F}_y \stackrel{\mathrm{def}}{=} \mathcal{F} \otimes F_y$ with the vacuum vector $|\mathrm{vac}\rangle^{\mathrm{tor}} \stackrel{\mathrm{def}}{=} |\mathrm{vac}\rangle \otimes 1$.

We now introduce the following operator acting on $\mathcal{F}_y \otimes \mathcal{F}_{y'}$:

$$\Omega^{\mathrm{tor}} \stackrel{\mathrm{def}}{=} \sum_{m\in\mathbb{Z}} \oint \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda}\, \psi(\lambda)V_m(\lambda; y) \otimes \psi^*(\lambda)V_{-m}(\lambda; y').$$

Using the anti-commutation relation (9) and the relation $V_m(y;\lambda)V_n(y;\lambda) = V_{m+n}(y;\lambda)$, we can obtain the following identity by direct calculations:

$$[\Omega^{\mathrm{tor}},\ \psi(p)\psi^*(p)V_n(y; p) \otimes 1 + 1 \otimes \psi(p)\psi^*(p)V_n(y'; p)] = 0$$

which means the action of $\mathfrak{sl}_2^{\mathrm{tor}}$ on $\mathcal{F}_y \otimes \mathcal{F}_{y'}$ commutes with Ω^{tor}. It is then straightforward to show that

$$\Omega^{\mathrm{tor}}(g|\mathrm{vac}\rangle^{\mathrm{tor}} \otimes g|\mathrm{vac}\rangle^{\mathrm{tor}}) = 0 \qquad (10)$$

for $g = \exp(X)$, $X \in \mathfrak{sl}_2^{\mathrm{tor}}$.

To rewrite (10) into bosonic language, we have the following two lemmas.

Lemma 1 ('Boson–fermion correspondence' [7,8]). *For any $|\nu\rangle \in \mathcal{F}$, we have the following formulae:*

$$\langle\mathrm{vac}|\psi_0^* \exp(H(\boldsymbol{x}))\psi(\lambda)|\nu\rangle = \exp(\xi(\boldsymbol{x},\lambda))\langle\mathrm{vac}|\exp(H(\boldsymbol{x} - [\lambda^{-1}]))|\nu\rangle$$
$$\langle\mathrm{vac}|\psi_{-1} \exp(H(\boldsymbol{x}))\psi^*(\lambda)|\nu\rangle = \lambda\exp(-\xi(\boldsymbol{x},\lambda))\langle\mathrm{vac}|\exp(H(\boldsymbol{x} + [\lambda^{-1}]))|\nu\rangle$$

where we have used the following notation:

$$\boldsymbol{x} = (x_1, x_3, \ldots)$$
$$H(\boldsymbol{x}) \stackrel{\mathrm{def}}{=} \sum_{n=1}^{\infty}\sum_{j\in\mathbb{Z}} x_n\psi_j\psi_{n+j}^*$$
$$\xi(\boldsymbol{x},\lambda) \stackrel{\mathrm{def}}{=} \sum_{n=1}^{\infty} x_n\lambda^n$$
$$[\lambda^{-1}] \stackrel{\mathrm{def}}{=} (1/\lambda, 1/2\lambda^2, 1/3\lambda^3, \ldots).$$

Lemma 2 ([9,10]). *Let $P(n) = \sum_{j\geqslant 0} n^j P_j$, where $P_j \in \mathrm{Diff}(z)$ are differential operators that may not depend on z. If*

$$\sum_{n\in\mathbb{Z}} z^n P(n)g(z) = 0$$

for some formal series $g(z) = \sum_j g_j z^j$, then

$$P(\epsilon - z\partial_z)g(z)|_{z=1} = 0$$

as a polynomial in ϵ.

Define the τ-function as

$$\tau(\boldsymbol{x},\boldsymbol{y}) \stackrel{\mathrm{def}}{=} {}^{\mathrm{tor}}\langle\mathrm{vac}|\exp(H(\boldsymbol{x}))g|\mathrm{vac}\rangle^{\mathrm{tor}}.$$

From relation (10), together with lemma 1 and 2, we have the following bilinear identity:

$$\oint \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\ \exp(\xi(\boldsymbol{x}-\boldsymbol{x}',\lambda))\tau(\boldsymbol{x} - [\lambda^{-1}], y_0 + \eta(\check{\boldsymbol{b}},\lambda^2), \check{\boldsymbol{y}} - \check{\boldsymbol{b}})$$
$$\times\tau(\boldsymbol{x}' + [\lambda^{-1}], y_0 - \eta(\check{\boldsymbol{b}},\lambda^2), \check{\boldsymbol{y}} + \check{\boldsymbol{b}}) = 0 \qquad (11)$$

where $\check{\boldsymbol{y}} \stackrel{\mathrm{def}}{=} (y_2, y_4, \ldots)$ and $\eta(\check{\boldsymbol{b}},\lambda^2) \stackrel{\mathrm{def}}{=} \sum_{n=1}^{\infty} b_{2n}\lambda^{2n}$.

Expanding (11), we can obtain Hirota-type differential equations including (2) and (3) ($x_1 = x,\ x_3 = t',\ y_0 = y,\ y_2 = t$). In this sense, the bilinear identity (11) is a generating function of Hirota-type differential equations of Bogoyavlensky's hierarchy. The N-soliton solution of (11) is obtained as follows [4]:

$$
\begin{aligned}
\tau_N(\boldsymbol{x},\boldsymbol{y}) &= \sum_{l=0}^{N}\sum_{j_1<\cdots<j_l} c_{j_1\cdots j_l}\prod_{m=1}^{l} a_{j_m}\exp(\eta_{j_m}(\boldsymbol{x},\boldsymbol{y}))\\
\eta_j(\boldsymbol{x},\boldsymbol{y}) &\stackrel{\text{def}}{=} \sum_{n=1}^{\infty} 2p_j^{2n-1}x_{2n-1} + \sum_{n=1}^{\infty} r_j p_j^{2n} y_{2n} \qquad (12)\\
c_{j_1\cdots j_l} &\stackrel{\text{def}}{=} \prod_{1\leqslant m<n\leqslant l}\frac{(p_{j_m}-p_{j_n})^2}{(p_{j_m}+p_{j_n})^2}.
\end{aligned}
$$

3. Derivation of the differential–difference system

We now apply the Miwa transformation [8, 14]

$$
\begin{aligned}
\boldsymbol{x}' &= l[\alpha] + m[\beta] + n[\gamma]\\
\boldsymbol{x} &= (l+1)[\alpha] + (m+1)[\beta] + (n+1)[\gamma]
\end{aligned} \qquad (13)
$$

to the bilinear identity (11). Here we have used the notation $l[\alpha] = (l\alpha, l\alpha^2/2, l\alpha^3/3, \ldots)$. We first consider the case $\check{\boldsymbol{b}} = (b_2, b_4, \ldots) = \boldsymbol{0}$. In this case, the bilinear identity (11) is reduced to that of the ordinary KP hierarchy. Thus we have the Hirota–Miwa equation (or the discrete KP equation)

$$
\begin{aligned}
&\alpha(\beta-\gamma)\ \tau(l+1,m,n;\boldsymbol{y})\tau(l,m+1,n+1;\boldsymbol{y})\\
&\quad+\beta(\gamma-\alpha)\ \tau(l,m+1,n;\boldsymbol{y})\tau(l+1,m,n+1;\boldsymbol{y})\\
&\quad+\gamma(\alpha-\beta)\ \tau(l,m,n+1;\boldsymbol{y})\tau(l+1,m+1,n;\boldsymbol{y}) = 0
\end{aligned} \qquad (14)
$$

where $\tau(l,m,n;\boldsymbol{y})$ denotes

$$\tau(l,m,n;\boldsymbol{y}) = \tau(\boldsymbol{x} = l[\alpha] + m[\beta] + n[\gamma], \boldsymbol{y}).$$

We then consider the time evolution with respect to y_0 and y_2. Collecting the coefficients of b_2 in the bilinear identity (11) gives

$$\oint \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\ \exp(\xi(\boldsymbol{x}-\boldsymbol{x}',\lambda))(D_{y_2} - \lambda^2 D_{y_0})\tau(\boldsymbol{x}'+[\lambda^{-1}],\boldsymbol{y})\cdot\tau(\boldsymbol{x}-[\lambda^{-1}],\boldsymbol{y}) = 0.$$

Applying the Miwa transformation (13) we obtain

$$
\begin{aligned}
&\alpha^2\beta\gamma(\beta-\gamma)D_{y_2}\tau(l+1,m,n;\boldsymbol{y})\cdot\tau(l,m+1,n+1;\boldsymbol{y})\\
&\quad+\alpha\beta^2\gamma(\gamma-\alpha)D_{y_2}\tau(l,m+1,n;\boldsymbol{y})\cdot\tau(l+1,m,n+1;\boldsymbol{y})\\
&\quad+\alpha\beta\gamma^2(\alpha-\beta)D_{y_2}\tau(l,m,n+1;\boldsymbol{y})\cdot\tau(l+1,m+1,n;\boldsymbol{y})\\
&= \beta\gamma(\beta-\gamma)D_{y_0}\tau(l+1,m,n;\boldsymbol{y})\cdot\tau(l,m+1,n+1;\boldsymbol{y})\\
&\quad+\gamma\alpha(\gamma-\alpha)D_{y_0}\tau(l,m+1,n;\boldsymbol{y})\cdot\tau(l+1,m,n+1;\boldsymbol{y})\\
&\quad+\alpha\beta(\alpha-\beta)D_{y_0}\tau(l,m,n+1;\boldsymbol{y})\cdot\tau(l+1,m+1,n;\boldsymbol{y})\\
&\quad-(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\\
&\quad\times D_{y_0}\tau(l,m,n;\boldsymbol{y})\cdot\tau(l+1,m+1,n+1;\boldsymbol{y}).
\end{aligned} \qquad (15)
$$

We further impose the condition $\beta = \gamma$. Then the τ-function $\tau(l,m,n;\boldsymbol{y})$ depends only on $k \stackrel{\text{def}}{=} m-n$, l and $\boldsymbol{y}$. In this sense, we rewrite

$$\tau(l,m,n;\boldsymbol{y}) \quad\rightarrow\quad \tau(l,k;\boldsymbol{y}).$$

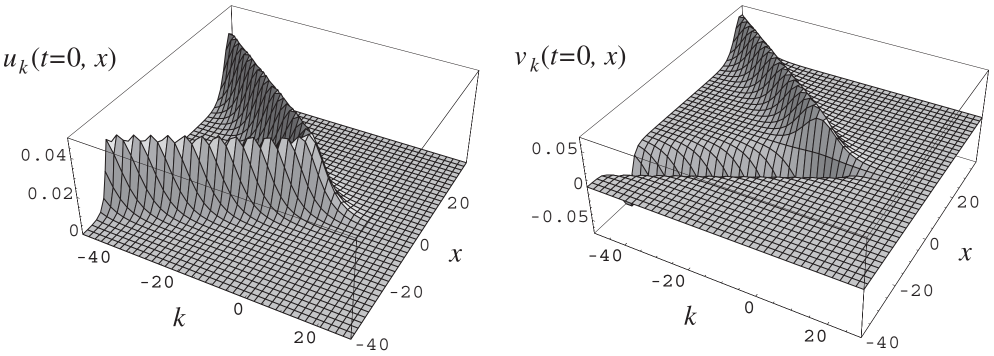

Figure 1. Special case of a two-soliton solution ('V-soliton') with $p_1 = p_2$ ($p_1 = p_2 = 0.3$, $r_1 = 0.15, r_2 = -0.1, a_1 = a_2 = 1, \alpha = 0.8, \beta = 0.5$).

Under this condition, (14) and (15) are reduced to

$$2\alpha\ \tau(l+1,k;\boldsymbol{y})\tau(l,k;\boldsymbol{y}) - (\alpha+\beta)\ \tau(l,k+1;\boldsymbol{y})\tau(l+1,k-1;\boldsymbol{y}) \\ -(\alpha-\beta)\ \tau(l,k-1;\boldsymbol{y})\tau(l+1,k+1;\boldsymbol{y}) = 0 \tag{16}$$

$$\alpha\beta^2 D_{y_2}(2\alpha\ \tau(l+1,k;\boldsymbol{y})\cdot\tau(l,k;\boldsymbol{y}) - (\alpha+\beta)\ \tau(l,k+1;\boldsymbol{y})\cdot\tau(l+1,k-1;\boldsymbol{y}) \\ -(\alpha-\beta)\ \tau(l,k-1;\boldsymbol{y})\cdot\tau(l+1,k+1;\boldsymbol{y})) \\ = D_{y_0}(2(2\beta^2-\alpha^2)\ \tau(l+1,k;\boldsymbol{y})\cdot\tau(l,k;\boldsymbol{y}) \\ -\alpha(\alpha+\beta)\ \tau(l,k+1;\boldsymbol{y})\cdot\tau(l+1,k-1;\boldsymbol{y}) \\ -\alpha(\alpha-\beta)\ \tau(l,k-1;\boldsymbol{y})\cdot\tau(l+1,k+1;\boldsymbol{y})). \tag{17}$$

Furthermore, we can construct the N-soliton solution by applying (13) to (12)

$$\tau_N(l,k;y_0,y_2) = \sum_{l=0}^{N}\sum_{j_1<\cdots<j_l} c_{j_1\cdots j_l}\prod_{m=1}^{l}\phi_{j_m}(l,k;y_0,y_2) \\ \phi_j(l,k;y_0,y_2) \stackrel{\text{def}}{=} a_j\exp(r_jy_0 + r_jp_j^2y_2)\left(\frac{1+p_j\alpha}{1-p_j\alpha}\right)^l\left(\frac{1+p_j\beta}{1-p_j\beta}\right)^k \tag{18}$$

where $c_{j_1\cdots j_l}$ is the same as the continuum one (12). We remark that the N-soliton τ-function can be written as the Wronskian determinant. Using the determinant expression, we can show that both (16) and (17) are reduced to the Plücker relations.

Introducing the variables as

$$\partial_t \stackrel{\text{def}}{=} \frac{2}{\alpha^2}\partial_{y_0} - 2\partial_{y_2} \qquad \partial_x \stackrel{\text{def}}{=} \partial_{y_2} - \frac{1}{\beta^2}\partial_{y_0}$$

$$u_k(t,x) \stackrel{\text{def}}{=} \log\left[\left(\frac{\beta-\alpha}{\beta+\alpha}\right)^{1/2}\frac{\tau(l+1,k+1)\tau(l,k)}{\tau(l,k+1)\tau(l+1,k)}\right]$$

$$v_k(t,x) \stackrel{\text{def}}{=} \partial_x\log\frac{\tau(l,k+2)}{\tau(l,k)}$$

we have the differential–difference equations (4) and (5), which have a N-soliton solution corresponding to the τ-function (18).

Let us consider the behaviour of $u_k(t,x)$ and $v_k(t,x)$ corresponds to (18) with $N = 2$ (two-soliton solution). The choice $p_1 = p_2$ gives a travelling wave solution that has a 'V' shape (figure 1). In generic cases ($p_1 \neq p_2$) the solution has the same features as the two-soliton solution of the KP equation (figure 2).

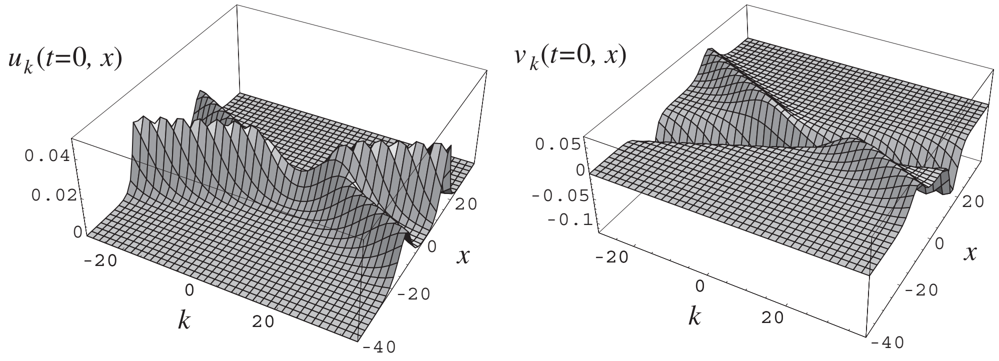

Figure 2. Two-soliton solution with $p_1 \neq p_2$ ($p_1 = 0.3$, $p_2 = 0.23$, $r_1 = 0.15$, $r_2 = -0.1$, $a_1 = a_2 = 1$, $\alpha = 0.8$, $\beta = 0.5$).

4. Concluding remarks

In this paper, we have introduced the differential–difference system (4) and (5), which is related to the toroidal Lie algebra, $\mathfrak{sl}_2^{\mathrm{tor}}$. Since the symmetry of the toroidal Lie algebra allows us to introduce extra parameters of wavenumbers in the soliton solution (i.e. r_j in (12) and (18)), it might be possible to construct some interesting solutions. In particular, we can obtain a class of travelling-wave solutions that are 'V' shaped (figure 1), which is a special case of two-soliton solutions. The existence of the V-soliton type solution is one of the features of this class of equations.

We note that there exist solutions of the same shape for the (2 + 1)-dimensional KdV equation (1), and for a (2 + 1)-dimensional generalization of the nonlinear Schrödinger (NLS) equation [15] that also has the symmetry of toroidal Lie algebra $\mathfrak{sl}_2^{\mathrm{tor}}$ [17]. We also remark that Oikawa *et al* [16] discussed the propagation of the V-soliton in a two-layer fluid, which is governed by an equation similar to the (2 + 1)-dimensional NLS equation.

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